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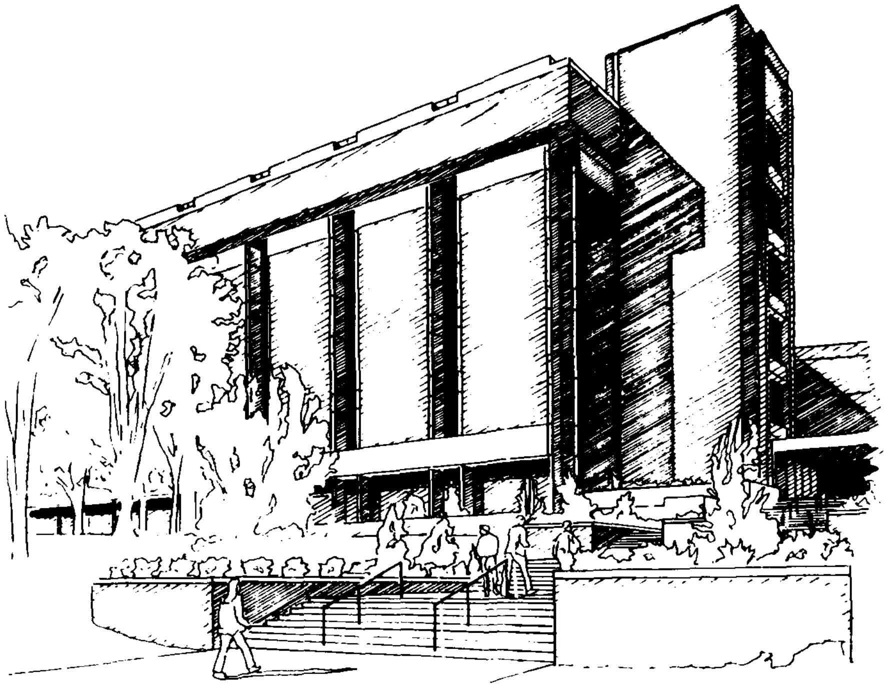

UNIVERSITY OF CINCINNATI
COLLEGE OF ENGINEERING

APPROXIMATE EVALUATION OF RELIABILITY AND RELATED QUANTITIES VIA PERTURBATION TECHNIQUES

Final Technical Report on
Grant AFOSR-88-0131

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ABSTRACT

Reliability evaluation of fault tolerant control systems (FTCS) that include sequential tests for failure detection and identification involve the transient analysis of finite-state semi-Markov chains of very large dimension. Such models for the time horizons of interest are intractable even for simple architectures. This has motivated the work summarized in this report as well as the work that was accomplished under a previous AFOSR grant.

The basis for the work is the idea of asymptotic aggregation of semi-Markov chains that include slow and fast transitions. The extension of earlier asymptotic aggregation results (primarily due to Korolyuk) to models that include decomposed classes that are non-ergodic and the subsequent application of these results to FTCS models was the subject of studies conducted under the previous grant.

The research efforts reported here concentrate on the application of the results from the previous study to more general FTCS architectures. A difficulty is pointed out in using the limit theorems of the previous work to approximate the occupancy probabilities of transient states that have large holding times. A modified algorithm that leads to better approximations for these states is presented in this report. The application of this modified algorithm to the nine state semi-Markov reliability model generated by a simple dual redundant system is then discussed.

Extension of the limit theorems developed in the earlier work to models with more general decomposed classes is also presented in this report. In particular, the results are extended to models that have multiple trapping states within each decomposed class.

1. INTRODUCTION

1.1 Motivation and Discussion of Problem

The work discussed in this report relates to the prediction of the reliability and other performance measures for complex fault tolerant systems. The criteria by which the design of a fault tolerant system is evaluated are usually related to predicted reliability and performance quantities. Thus, the development of the means for making such calculations has become a major concern as the reliability and performance requirements of these systems have been made more demanding.

One method that has proven to be useful for evaluating the reliability and performance of a fault tolerant system is the use of finite state Markov and semi-Markov models [1-6]. As the introduction to the final report on the previous grant [7] points out, however, four basic difficulties to the numerical implementation arise when generalized Markovian models are used to represent the behavior of a fault tolerant system:

1. The dimension of the model is typically large, even for relatively simple architectures.

2. The transient behavior of the model is the behavior of interest because the operating time of the system is always much shorter than the time necessary for all the components fail (which is the steady state condition). Generating numerical results for the transient behavior of generalized Markovian models is a problem for which few short-cuts exist.

3. Although the operating times of interest are short relative to the time necessary for all of the components to fail, it is often long relative to the time between applications of the failure detection tests that are in use. Thus, we are often interested in the transient behavior of a Markovian model after many thousands of time steps.

4. A significant time scale separation exists between the average times necessary for a failure to occur and the average time necessary for the failure detection decisions to be made. However, some failure detection decisions, particularly "false alarm" decisions (a decision that a failure is present when in fact one is not present), require an average time of considerable length relative to the component mean time to failure. These difficulties make the problem of evaluating such quantities as the reliability of a fault tolerant control system impractical even with powerful digital computing machinery. It is this intractability with which this study (and the previous study) is concerned.

1.2 Summary of Previous Work

The four difficulties discussed in the preceding subsection motivate the development of approximate strategies to quantitatively solve the Markovian models that represent fault tolerant systems. Many of the approximate techniques (including those discussed in [7] and in this report) are based upon the exploitation of the time scale separation mentioned in Comment 4 above. The basic idea is to aggregate into classes the states of the model that share transitions that occur in the fast time scale. Then, the transitions between the classes can be characterized by the slow time scale behavior while the behavior within each class is assumed to be well-approximated by the steady state behavior of the class with slow transitions neglected. This leads to two types of Markovian models to be solved, each of relatively small dimension: a transient model that evolves in the slow time scale describing the interclass transition dynamics, and a steady state model for each class. The approximate behavior of the original model over long time horizons is then approximated by multiplying the transient probability of occupying each class by the steady state

probabilities of occupying each state within that class.

The introduction of [7] summarizes much of the work done by other authors on the problem of aggregating finite state Markovian models to generate approximate solutions. In the interest of brevity, that summary is not repeated here. The interested reader is referred to [7].

The primary contribution of the work reported in [7] was to extend some previous results (primarily due to Korolyuk [8,9]) to situations that more accurately reflect the types of models that represent the behavior of fault-tolerant systems. In particular, the work summarized in [7]:

- Extended Korolyuk's theorems to time-scaled models that do not decompose into ergodic decomposed classes.
- Developed the discrete time versions of Korolyuk's original theorems and the discrete time analog of the extension to models with nonergodic decomposed classes.
- Applied the discrete time results to a relevant, though simple, discrete time fualt tolerant system model for which exact results could also be generated.
- Made a preliminary effort to extend the results further to models that include decomposed classes with multiple trapping states.

As was pointed out in [7], these contributions make feasible the use of semi-Markov reliability models as a design tool for fault tolerant systems by making it possible to approximately solve for the reliability and other performance measures efficiently.

1.3 Research Goals of Present Work

The work reported here was motivated by two shortcomings of the work reported in [7]. One is alluded to in the contributions summary of the previous subsection. The other relates to the assumption that essentially

all approximate aggregation strategies make regarding the strong separation in time scales of the slow and fast transition behavior.

In [7], an incomplete proof was given for applying the approximate aggregation results to models that yield multiple trapping states in the decomposed classes. One goal of the work reported here was to complete this proof.

The other shortcoming in the results of [7] relates to the numerical results that were obtained for a model that represented the behavior of a simple dual redundant system used in a primary-secondary mode (cf. Section 2.5.2 of [7]). If a decomposed class of the model contains at least one trapping state when slow transitions are neglected, then the steady state distribution within that class will indicate zero probability for any state that is transient. Hence, when the approximation is constructed, the approximate solution predicts zero probability for these states. This is not a good approximation if these states have average holding times that are a significant fraction of the time scale over which the slow transitions occur, as would be the case for a fault tolerant system with very long mean times to false alarm. Much of the work reported below is an effort to address this problem.

2. PROGRESS SUMMARY

2.1 Summary of Section

This section summarizes the salient aspects of the work carried out during the one-year period of the grant. Section 2.2 describes the extension of earlier limit theorems for semi-Markov chains to more general case that includes multiple recurrent subsets in decomposed classes. Derivation of the semi-Markov model for a simple dual redundant system is presented in section 2.3. Application of the earlier limit theorem and a modified algorithm developed during the period of this research on the resulting nine state model is then discussed in a paper that is included as Appendix A and summarized in section 2.4.

2.2. Behavioral Decomposition of Semi-Markov Models - General Case

The decomposition of semi-Markovian models that include fast and slow transitions between the various states was considered by Korolyuk [8]. It was shown in [8] that a semi-Markov chain that depended on a small parameter ε which could be split in the limit as ε 0 into a disjoint set of non-communicating classes of states E_k, k=1,...,m, can be represented by a m-state Markov chain when the classes are ergodic. Extension of this result to non-ergodic classes E_k that included one irreducible recurrent subset of states was given in [7]. We consider below more general classes E_k and extend the limit theorem for semi-Markov processes originally considered in [8].

Let the set of states of a semi-Markov process that depends on a small parameter ε be split into disjoint classes of states,

$$E = \sum_{k=1}^{m} E_k \qquad (2.1)$$

such that the probability of departure from each class and the sojourn time

in a given state tend to zero along with ε. The total sojourn in each class is assumed to have a nondegenerate distribution in the limit as ε 0. Further, let each of the classes be split into a set of transient states E_{k_t} and irreducible recurrent subsets E_{k_l} such that,

$$E_k = E_{k_t} + \sum_{l=1}^{n} E_{k_l} \tag{2.2}$$

where n may in general be different for various classes. Note that the recurrent assumption on the subsets E_{k_l} implies that these subsets are ergodic. We derive below the nature of the inter-class transitions when ε tends to zero.

Let the elements of the transition probability matrix $(P^{\varepsilon}_{ij}(t);\ i,j \in E)$ specifying the semi-Markov process depend as follows on the small parameter ε:

$$P^{\varepsilon}_{ij}(t) = p^{\varepsilon}_{ij}\, F_{ij}(t/\varepsilon)\ ; \quad i,j \in E \tag{2.3}$$

with,

$$p^{\varepsilon}_{ij} = \begin{cases} p^{(k)}_{ij} - \varepsilon q^{(k)}_{ij}\ ; & i,j \in E_k \\ \varepsilon q^{(k)}_{ij}\ ; & i \in E_k\ ,\ j \notin E_k \end{cases} \tag{2.4}$$

Here,

$$\sum_{j\in E_k} p^{(k)}_{ij} = 1\ ,\ i \in E_k \tag{2.5}$$

Also, since each E_{k_l} represents an irreducible recurrent subset of states, we have

$$\sum_{j\in E_{k_l}} p^{(k_l)}_{ij} = 1\ ,\ i \in E_{k_l} \tag{2.6}$$

We further make the assumption that all out of class transitions when ε 0 take place only from the ergodic states in each class. Let $\tau^{(i)}_{k_l r_v}$ be the sojourn of the semi-Markov process in the subset E_{k_l} of class k when it

starts from the state $i \in E_{k_l}$ and moves to the irreducible recurrent subset E_{r_v} in class r. The transitions from the subset E_{k_l} to E_{r_v} can be split into direct transitions and indirect transitions through the transient set E_{r_t} in class r such that:

$$P\left\{\tau^{(i)}_{k_l r_v} \le t\right\} = P\left\{\tau^{(i)d}_{k_l r_v} \le t\right\} + P\left\{\tau^{(i)t}_{k_l r_v} \le t\right\} \qquad (2.7)$$

Here, the first term on the right hand side of (2.7) represents the direct transition sojourn and the second term the sojourn for a transition through the transient set in class r.

Let $\varepsilon\zeta_{ij}$ denote the sojourn of the semi-Markov process in the i-th state with distribution $F_{ij}(t)$, while $\delta^{\varepsilon}_{ij}$ are the indicators of transition from the i-th to the j-th state. Using the expression for total probability, we obtain for the random quantities $\tau^{(i)d}_{k_l r_v}$:

$$P\left\{\tau^{(i)d}_{k_l r_v} \le t\right\} = \sum_{j \in E_{k_l}} P\left\{\delta^{\varepsilon}_{ij}=1,\ \varepsilon\zeta_{ij} + \tau^{(j)d}_{k_l r_v} \le t\right\}$$

$$+ \sum_{j \in E_{r_v}} P\left\{\delta^{\varepsilon}_{ij}=1,\ \varepsilon\zeta_{ij} \le t\right\} \qquad (2.8)$$

Since subsets E_{k_l} and E_{r_v} contain only ergodic states by construction, we have from Korolyuk's limit theorem [8] :

$$\lim_{\varepsilon\ 0} P\left\{\tau^{d}_{k_l r_v} \le t\right\} = p^{d}_{k_l r_v} \left(1 - \exp(-\Lambda_{kl} t)\right) \qquad (2.9)$$

which is independent of i. Here,

$$p^{d}_{k_l r_a} = \frac{\sum_{i \in E_{k_l}} \pi^{(k_l)}_i q^{(k_l r_v)}_i}{\sum_{i \in E_{k_l}} \pi^{(k_l)}_i q^{(k_l)}_i} \ ; \quad \Lambda_{kl} = \frac{\sum_{i \in E_{k_l}} \pi^{(k_l)} q^{(k_l)}_i}{\sum_{i \in E_{k_l}} \pi^{(k_l)}_i a^{(k_l)}_i} \qquad (2.10)$$

$$q_i^{(k_l r_v)} = \sum_{j \in E_{r_v}} q_{ij}^{(k_l)} \; ; \qquad q_i^{(k_l)} = \sum_{j \in E_{k_l}} q_{ij}^{(k_l)} \tag{2.11}$$

$$a_i^{(k_l)} = \sum_{j \in E_{k_l}} a_{ij} p_{ij}^{(k_l)} \; ; \qquad a_{ij} = \int_0^\infty tF_{ij}(t) \tag{2.12}$$

In the above, $\pi_i^{(k_l)}$ represents the stationary probability distribution in the imbedded Markov-chain defined by $p_{ij}^{(k_l)}$. The elements $p_{ij}^{(k_l)}$ and $q_{ij}^{(k_l)}$ are respectively the $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ terms for the indices $i, j \in E_{k_l}$.

The probability distribution of the sojourn time for transition from E_{k_l} to E_{r_v} through the transient set E_{r_t} can be further split into:

$$P\left\{\tau_{k_l r_v}^{(i)t} \le t\right\} = p_{r_t r_v} P\left\{\tau_{k_l r_t}^{(i)} \le t\right\} \tag{2.13}$$

Here, the term $p_{r_t r_v}$ represents the probability of transition from the transient set E_{r_t} to the subset E_{r_v} within class r and is given by:

$$p_{r_t r_v} = \frac{\sum_{i \in E_{r_t}} \sum_{j \in E_{r_v}} p_{ij}^{(r)}}{\sum_v \sum_{i \in E_{r_t}} \sum_{j \in E_{r_v}} p_{ij}^{(r)}} \tag{2.14}$$

Proceeding as before with similar notation, we obtain for the distribution of sojourn time starting from the i-th state in E_{k_l} to the j-th state in E_{r_t}:

$$P\left\{\tau_{k_l r_t}^{(i)} \le t\right\} = \sum_{j \in E_{k_l}} P\left\{\delta_{ij}^{\varepsilon}=1,\ \varepsilon\zeta_{ij} + \tau_{k_l r_t}^{(j)} \le t\right\}$$

$$+ \sum_{j \in E_{r_t}} P\left\{\delta_{ij}^{\varepsilon}=1,\ \varepsilon\zeta_{ij} \le t\right\} \tag{2.15}$$

Hence,

$$P\left\{\tau^{(i)}_{k_l r_t} \le t\right\} = \sum_{j\in E_{k_l}} \int_0^t P\left\{\tau^{(i)}_{k_l r_t} \le t-u\right\} dP^{\varepsilon}_{ij}(u)$$

$$+ \sum_{j\in E_{r_t}} P^{\varepsilon}_{ij}(t) \tag{2.16}$$

Taking the Laplace transform of (2.16), we obtain

$$\varphi^{(i)}_{k_l r_t}(s) = \sum_{j\in E_{k_l}} \varphi^{(j)}_{k_l r_t}(s)\, p^{\varepsilon}_{ij}(s) + \sum_{j\in E_{r_t}} \frac{1}{s}\, p^{\varepsilon}_{ij}(s) \tag{2.17}$$

where,

$$p^{\varepsilon}_{ij}(s) = \int_0^{\infty} e^{-st}\, dP^{\varepsilon}_{ij}(t) \tag{2.18}$$

Recalling (2.3) and (2.4), after simple manipulations it can be shown that:

$$p^{\varepsilon}_{ij}(s) = \left(p^{(k_l)}_{ij} - \varepsilon q^{(k_l)}_{ij}\right)\cdot\left(1 - \varepsilon s a_{ij}\right) \quad ; \; j\in E_{k_l} \tag{2.19}$$

$$p^{\varepsilon}_{ij}(s) = \varepsilon q^{(k_l)}_{ij} \quad ; \; j\in E_{r_t} \tag{2.20}$$

Substituting these expressions in (2.17), we get:

$$\varphi^{(i)}_{k_l r_t}(s) - \sum_{j\in E_{k_l}} \varphi^{(j)}_{k_l r_t}(s)\, p^{(k_l)}_{ij} = \varepsilon \sum_{j\in E_{r_t}} \frac{1}{s}\, q^{(k_l)}_{ij}$$

$$- \varepsilon \sum_{j\in E_{k_l}} \left(s a_{ij} p^{(k_l)}_{ij} + q^{(k_l)}_{ij}\right) \varphi^{(j)}_{k_l r_t}(s) \tag{2.21}$$

Passing to the limit as ε 0, the functions $\varphi^{(i)}_{k_l r_t}(s)$ are found to satisfy the system of equations:

$$\varphi^{(i)}_{k_l r_t}(s) - \sum_{j\in E_{k_l}} \varphi^{(j)}_{k_l r_t}(s)\, p^{(k_l)}_{ij} = 0 \tag{2.22}$$

It follows from this and the fact that the Markov chain defined by the transition probabilities $p^{(k_l)}_{ij}$, $i, j\in E_{k_l}$ is ergodic, that the solution of the

system is independent of the superscript, i.e. $\varphi^{(i)}_{k_l r_t}(s) = \varphi_{k_l r_t}(s)\ \forall\ i \in E_{k_l}$.

Multiplying (2.21) by the stationary probabilities $\pi_i^{(k_l)}$ and summing over all $i \in E_{k_l}$ and cancelling ε, we get:

$$\sum_{i \in E_{k_l}} \pi_i^{(k_l)} \sum_{j \in E_{k_l}} \left(s a_{ij} p_{ij}^{(k_l)} + q_{ij}^{(k_l)} \right) \varphi^{(j)}_{k_l r_t}(s) =$$

$$\sum_{i \in E_{k_l}} \pi_i^{(k_l)} \sum_{j \in E_{r_t}} q_{ij}^{(k_l)} \tag{2.23}$$

On passing to the limit ε 0, noting that all the $\varphi^{(i)}_{k_l r_t}(s)$ have the limit function $\varphi_{k_l r_t}(s)$, we obtain,

$$\varphi_{k_l r_t}(s) = p_{k_l r_t} \frac{1}{s} \frac{\Lambda_{kl}}{s + \Lambda_{kl}} \tag{2.24}$$

which is independent of the starting state i in E_{k_l}. Taking the inverse transform,

$$\lim_{\varepsilon\ 0} P\left\{ \tau_{k_l r_t} \le t \right\} = p_{k_l r_t} \left(1 - \exp(-\Lambda_{kl} t) \right) \tag{2.25}$$

Here, Λ_{kl} is same as defined in (2.10) and,

$$p_{k_l r_t} = \frac{\sum_{i \in E_{k_l}} \pi_i^{(k_l)} q_i^{(k_l r_t)}}{\sum_{i \in E_{k_l}} \pi_i^{(k_l)} q_i^{(k_l)}} \tag{2.26}$$

with,

$$q_i^{(k_l r_t)} = \sum_{j \in E_{r_t}} q_{ij}^{(k_l)} \tag{2.27}$$

Hence, the probability distribution of the sojourn time from the set E_{k_l} to E_{r_v} is independent of the starting state and is given by:

$$\lim_{\varepsilon \to 0} P\left\{\tau_{k_l r_v} \le t\right\} = \left(p^{d}_{k_l r_v} + p_{k_l r_t}\, p_{r_t r_v}\right) * \left(1 - \exp(-\Lambda_{kl} t)\right) \qquad (2.28)$$

This is a general result that is valid for any arbitrary set of classes defined by (2.1). In this sense then, this result is an extension of the results reported in [7]. In fact, it generalizes the approximate aggregation results to essentially all fault tolerant system models in semi-Markov form.

2.3. Model of dual redundant system

In this section, a semi-markov model is developed that represents the behavior of a simple dual redundant system. This model is identical to a model considered in [7]. However, the construction of the model presented here differs from the model construction method employed in [7] and therefore warrants detailed presentation. The next section will discuss the numerical results generated for this model.

Consider a dual redundant fault tolerant system architecture where two identical instruments are measuring a scalar quantity of interest. An instrument is deemed to have failed when the measurements are corrupted by a bias error sufficiently large to cause mission failure if the output is used in the control scheme. At the two level stage where both instruments are operational, two one-sided sequential ratio detection tests (SRDT) are used to detect a failure [7]. If the SRDTs simultaneously arrive at a decision, the test is reset. Depending on whether SRDT 1 or SRDT 2 arrives at a decision, an isolation option of either 1 or 2 is triggered. In either option, two sequential probability ratio tests (SPRT) are used to arrive at an isolation decision or an alarm rejection as follows:

Isolation Option 1:

SPRT 1	SPRT 2	Decision
Isolation	Isolation	Reinitiate SPRT's
Isolation	Rejection	Declare instrument 1 failed
Rejection	Isolation	Declare instrument 2 failed
Rejection	Rejection	Reject SRDT alarm

Similar isolation options following the triggering of SRDT 2 are defined.

Once an instrument is declared faulty, the FDI tests are discontinued. To simplify the calculation of the exact results required for comparison to the approximate results that will be generated later, it is assumed that all tests are reset after five time samples, which reduces the number of terms in the numerical convolution sums to be evaluated to a maximum of five. For isolation, we assume that an independent estimate of the scalar quantity of interest is available to vote between the two instruments.

For developing the semi-Markov model for this fault tolerant architecture, it is suggested in [7] that all the states of the model must be enumerated first, based on the FDI logic and failure event occurence. Another approach is to start with a state characterized by all working components and then list the various combinatorial events that can happen for this state during a single time step. This method is more efficient and does not require the states of the model to be enumerated apriori. After considering the various combinatorial events and hence the possible states for the model, we derive the core matrix elements [9] for the semi-Markov model. The core matrix elements are then examined to see whether they represent a valid probability mass function. If the probability masses for transitions out of a particular state does not sum to one, then it implies either there are other states to which it can transition to, or the core

matrix elements for transitions from this state are in error. We then suitably modify the model and check for a valid probability mass function. Thus, the construction of the semi-Markov model in general becomes a relatively simple trial and error procedure.

We start the model development for the two component redundant system from a state that rpresents a condition with all components available, none failed and no detection alarm present. The various possible events that can occur during one time step are as follows:

(1) No failure but SRDT alarm occurs.

(2) One instrument failed and correct detection alarm occurs.

(3) One instrument failed, detection alarm occurs for wrong pair.

(4) One instrument failed and no detection alarm occurs.

Thus by considering all possible combinatorial events from a known state, we have enumerated four other states of the model. We then proceed as above to list other possible states for the model from these known states.

For this FDI scheme with the given architecture, we get a nine state model. The nine states and the notation used to represent them are given below.

1. *Two instruments available, none failed, no detection alarm present, SRDT's operating. (2/0/0)*
2. *Two instruments available, none failed, one SRDT detection alarm present, SPRT's operating. (2/0/D)*
3. *One instrument available, one eliminated due to false isolation [FDI discontinued]. (1G/FI)*
4. *Two instruments available, one failed, correct detection alarm triggered, SPRT's operating. (2/F/C)*
5. *Two instruments available, one failed, detection alarm present*

for wrong pair, SPRT's operating. (2/F/W)

6. *Two instruments available, one failed, no detection alarms present, SRDT's operating. (2/F/0)*
7. *One good instrument available, one faulty instrument isolated [FDI discontinued]. (1G/F)*
8. *System loss due to one failure and one false isolation. (SL/F/FI)*
9. *System loss due to two failures. (SL/2F)*

The state transition diagram for this model is given in Appendix A. The thick lines indicate fast transitions while the dashed lines indicate slow transitions. The self loops in each state indicate the reset mechanism incorporated in the tests. We notice that some of the states in this model can be aggregated, for instance states 8 and 9 which represent a vehicle loss. This has not been done in order that when the nine state model is decomposed, it breaks up into three classes with the inter-class transitions taking place in the slow time-scale (of order ε, the failure rate of the components).

2.3.1. Set notation used

Before deriving the core matrix elements we present the set notation used to represent the various events associated with the model.

F^i_m A failure of instrument i at time sample m

$\bar{F}^i_m$ No failure of instrument i at time sample m

D^i_m A failure indication by SRDT for instrument i at time sample m

$\bar{D}^i_m$ No decision available from SRDT on instrument i at time sample m

R^i_m Rejection of alarm by SPRT for instrument i at time

sample m

I_m^i Isolation decision by SPRT for instrument i at time sample m

$\cap$ Denotes the intersection of events

$\cup$ Denotes the union of events

It is assumed in the development of the semi-Markov model that all of the events are statistically independent. A complete statistical description of the SRDT and SPRT used in the FDI scheme requires knowledge of the conditional probability mass functions (pmf) of the time to decision of the sequential tests. The following functions are used to describe the behavior of the sequential tests.

$f_D^0(\cdot)$ pmf for detection given no failure is present for SRDT

$f_D^1(\cdot)$ pmf for detection given failure is present for SRDT

$f_I^0(\cdot)$ pmf for detection given no failure is present for SPRT

$f_R^0(\cdot)$ pmf for rejection given no failure is present for SPRT

$f_I^1(\cdot)$ pmf for detection given failure is present for SPRT

$f_R^1(\cdot)$ pmf for rejection given failure is present for SPRT

The above pmf's can be expressed in the event notation as follows:

$$f_D^0(m) = \Pr\left\{D_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^i\right\} \tag{2.29}$$

$$f_D^1(m) = \Pr\left\{D_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; F_0^i\right\} \tag{2.30}$$

$$f_R^0(m) = \Pr\left\{R_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^i\right\} \tag{2.31}$$

$$f_I^0(m) = \Pr\left\{I_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^i\right\} \tag{2.32}$$

$$f_R^1(m) = \Pr\left\{R_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; F_0^i\right\} \tag{2.33}$$

$$f_I^1(m) = \Pr\left\{ I_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; F_0^i \right\} \tag{2.34}$$

An important observation to be made concerning the above *pmf*'s is that the fault monitoring event at time sample m is conditioned on the failure events that take place prior to and including time m-1. Thus, it is assumed that there is a delay of at least a single time step between when a failure takes place and when it is detected.

We further define the quantities $S_i^j(m)$, which will be used frequently in the derivation of the core matrix elements.

$S_0^1(m)$ Probability that no decision has been reached at a given time m in the absence of a failure for SRDT

$S_1^1(m)$ Probability that no decision has been reached at a given time m in the presence of a failure for SRDT

Identical quantities for the SPRT are defined with the superscripts taking the value 2 in each case.

In terms of the event notation each of the above quantities can be expressed as follows:

$$S_0^1(m) = \Pr\left\{ \bar{D}_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^i \right\} \; ; \; m\geq 1 \tag{2.35a}$$

$$S_0^1(m) = 1 - \sum_{k=1}^{m} f_D^0(k) \; ; \; m\geq 1 \tag{2.35b}$$

$$S_1^1(m) = \Pr\left\{ \bar{D}_m^i \bigcap_{k=1}^{m-1} \bar{D}_m^i \;\middle|\; F_0^i \right\} \; ; \; m\geq 1 \tag{2.36a}$$

$$S_1^1(m) = 1 - \sum_{k=1}^{m} f_D^1(k) \; ; \; m\geq 1 \tag{2.36b}$$

$$S_0^2(m) = 1 - \sum_{k=1}^{m} \left[f_R^0(k) + f_I^0(k) \right] \; ; \; m\geq 1 \tag{2.37}$$

$$S_1^2(m) = 1 - \sum_{k=1}^{m} \left[f_R^1(k) + f_I^1(k) \right] \; ; \; m \geq 1 \tag{2.38}$$

An additional assumption made in deriving the model is that the failures exhibit a geometrically distributed time of occurence. The probability of failure ε over a single time step can be expressed as:

$$\Pr\left\{ F_m \,\middle|\, \bar{F}_{m-1} \right\} = \varepsilon \tag{2.39}$$

2.3.2. Core matrix element derivation

We now proceed to derive the core matrix elements for the semi-Markov model of the dual redundant system. To be concise, we indicate the steps involved in the derivation of the core matrix element $g_{ij}(\cdot)$ for one case and give the final expressions for other cases.

Let us examine in detail the core matrix element $g_{11}(m)$, i.e., a reset of state 1. A reset arises at time m if no failures take place for all times prior to and including m, the chain entered state 1 at m=0, and both SRDT's indicate a detection of failure at time m. Since it is assumed that the tests are reset after m=5, we have:

$$g_{11}(m) = \Pr\left\{ \left[\bigcap_{k=1}^{m} \bar{D}_k^1 \, \bar{D}_k^2 \right] \left[\bigcap_{k=1}^{m} \bar{F}_k^1 \, \bar{F}_k^2 \right] \right\} \delta_{m5}$$
$$+ \Pr\left\{ \left[D_m^1 \, D_m^2 \right] \left[\bigcap_{k=1}^{m-1} \bar{D}_k^1 \, \bar{D}_k^2 \right] \left[\bigcap_{k=1}^{m} \bar{F}_k^1 \, \bar{F}_k^2 \right] \right\} \tag{2.40}$$

Noting that all of the above events are conditionally independent,

$$g_{11}(m) = \Pr\left\{ \bigcap_{k=1}^{m} \bar{F}_k^1 \right\} \Pr\left\{ \bigcap_{k=1}^{m} \bar{F}_k^2 \right\} \Pr\left\{ \bigcap_{k=1}^{m} \bar{D}_k^1 \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^1 \right\}$$

$$\Pr\left\{ \bigcap_{k=1}^{m} \bar{D}_k^2 \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^2 \right\} \delta_{m5}$$

$$+ \Pr\left\{ \bigcap_{k=1}^{m} \bar{F}_k^1 \right\} \Pr\left\{ \bigcap_{k=1}^{m} \bar{F}_k^2 \right\} \Pr\left\{ D_m^1 \bigcap_{k=1}^{m-1} \bar{D}_k^1 \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^1 \right\}$$

$$\Pr\left\{ D_m^2 \bigcap_{k=1}^{m-1} \bar{D}_k^2 \;\middle|\; \bigcap_{k=1}^{m-1} \bar{F}_k^2 \right\} \qquad (2.41)$$

Applying the definitions for the pmf and using the notation defined earlier, we can simplify (2.41) to yield:

$$g_{11}(m) = (1-\varepsilon)^{2m} \left[1 - \sum_{k=1}^{m} f_D^0(k)\right]^2 \delta_{m5} + (1-\varepsilon)^{2m} \left[f_D^0(m)\right]^2 \qquad (2.42)$$

Consider the core matrix element $g_{21}(m)$, which is the pmf for the SPRTs to arrive at a no failure decision given no failure is present. This transition occurs when either SPRT arrives at a no failure decision at any time prior to and including m and the the other SPRT decides on a no failure decision at instant m given that no failures are present. This can be represented in the event notation as,

$$g_{21}(m) = \Pr\left\{ \left(R_m^1 \; R_m^2 \right) \left[\bigcap_{k=1}^{m-1} \bar{D}_k^1 \; \bar{D}_k^2 \right] \left[\bigcap_{k=1}^{m} \bar{F}_k^1 \; \bar{F}_k^2 \right] \right\}$$

$$+ 2\Pr\left\{ \left(R_m^1 \bigcap_{k=1}^{m-1} \hat{D}_k^1 \right) \left[\sum_{k=1}^{m-1} \bigcap_{j=1}^{m-1} R_j^2 \; \hat{D}_{j-1}^2 \right] \left[\bigcap_{k=1}^{m} \bar{F}_k^1 \; \bar{F}_k^2 \right] \right\} \qquad (2.43)$$

Using the conditional independence of various events this simplifies to,

$$g_{21}(m) = (1-\varepsilon)^{2m} \left[\left(f_R^0(m) \right)^2 + 2\, f_R^0(m) \sum_{k=1}^{m-1} f_R^0(k) \right] \qquad (2.44)$$

Proceeding along similar lines, the core matrix elements for other cases are summarized below.

$$g_{12}(m) = 2(1-\varepsilon)^{2m}\, f_D^0(m)\left[1 - \sum_{k=1}^{m} f_D^0(k)\right] \tag{2.45}$$

$$g_{14}(m) = 2\varepsilon(1-\varepsilon)^{(2m-1)} f_D^0(m)\left[1 - \sum_{k=1}^{m} f_D^0(k)\right] \tag{2.46}$$

$$g_{15}(m) = g_{14}(m) \tag{2.47}$$

$$g_{16}(m) = 2\varepsilon(1-\varepsilon)^{(2m-1)}\left[1 - \sum_{k=1}^{m} f_D^0(k)\right]^2 \tag{2.48}$$

$$g_{1i}(m) = 0 \;;\; i=3,7,8,9 \tag{2.49}$$

$$g_{22}(m) = (1-\varepsilon)^{2m}\left[\left(f_I^0(m)\right)^2 + 2\, f_I^0(m) \sum_{k=1}^{m-1} f_I^0(k)\right]$$

$$+\, 2(1-\varepsilon)^{2m}\left[1 - S_0^2(m)\right]\left[f_I^0(m) + f_R^0(m)\right]\delta_{m5} \tag{2.50}$$

$$g_{23}(m) = 2(1-\varepsilon)^{2m}\left[f_I^0(m) \sum_{k=1}^{m} f_R^0(k) + f_R^0(m) \sum_{k=1}^{m-1} f_I^0(k)\right] \tag{2.51}$$

$$g_{24}(m) = 0.5(1-\varepsilon)^{(2m-1)} \tag{2.52}$$

$$g_{25}(m) = g_{24}(m) \tag{2.53}$$

$$g_{26}(m) = \varepsilon(1-\varepsilon)^{(2m-1)}\left[\left(f_R^0(m)\right)^2 + 2\, f_R^0(m) \sum_{k=1}^{m-1} f_R^0(k)\right] \tag{2.54}$$

$$g_{2i}(m) = 0 \;;\; i=7,8,9 \tag{2.55}$$

$$g_{33}(m) = (1-\varepsilon)\delta_{m1} \tag{2.56}$$

$$g_{38}(m) = \varepsilon\, \delta_{m1} \tag{2.57}$$

$$g_{3i}(m) = 0 \;;\; i\neq 3,8 \tag{2.58}$$

$$g_{44}(m) = (1-\varepsilon)^m \left[f_I^1(m) \sum_{k=1}^{m} f_I^0(k) + f_I^0(m) \sum_{k=1}^{m-1} f_I^1(k) \right]$$

$$+ (1-\varepsilon)^m \left\{ S_0^2(m) \left[1-S_1^2(m)\right] + S_1^2(m) \left[1-S_0^2(m)\right] \right\} \delta_{m5} \qquad (2.59)$$

$$g_{46}(m) = (1-\varepsilon)^m \left[f_R^1(m) \sum_{k=1}^{m} f_R^0(k) + f_R^0(m) \sum_{k=1}^{m-1} f_R^1(k) \right] \qquad (2.60)$$

$$g_{47}(m) = (1-\varepsilon)^m \left[f_I^1(m) \sum_{k=1}^{m} f_R^0(k) + f_R^0(m) \sum_{k=1}^{m-1} f_I^1(k) \right] \qquad (2.61)$$

$$g_{48}(m) = (1-\varepsilon)^m \left[f_R^1(m) \sum_{k=1}^{m} f_I^0(k) + f_I^0(m) \sum_{k=1}^{m-1} f_R^1(k) \right] \qquad (2.62)$$

$$g_{49}(m) = \varepsilon(1-\varepsilon)^{(m-1)} \left\{ S_0^2(m) \left[1-S_1^2(m)\right] + S_1^2(m) \left[1-S_0^2(m)\right] \right\} \qquad (2.63)$$

$$g_{4i}(m) = 0 \; ; \; i=1,2,3,5 \qquad (2.64)$$

$$g_{55}(m) = (1-\varepsilon)^m \left[f_I^0(m) \, f_I^0(m) + 2 \, f_I^0(m) \sum_{k=1}^{m-1} f_I^0(k) \right]$$

$$+ 2(1-\varepsilon)^m \, S_0^2(m) \left[1-S_0^2(m)\right] \delta_{m5} \qquad (2.65)$$

$$g_{56}(m) = (1-\varepsilon)^m \left[f_R^0(m) \, f_R^0(m) + 2 \, f_I^0(m) \sum_{k=1}^{m-1} f_I^0(k) \right] \qquad (2.66)$$

$$g_{57}(m) = (1-\varepsilon)^m \left[f_I^0(m) \sum_{k=1}^{m} f_R^0(k) + f_R^0(m) \sum_{k=1}^{m-1} f_I^0(k) \right] \qquad (2.67)$$

$$g_{58}(m) = (1-\varepsilon)^m \left[f_R^0(m) \sum_{k=1}^{m} f_I^0(k) + f_I^0(m) \sum_{k=1}^{m-1} f_R^0(k) \right] \qquad (2.68)$$

$$g_{59}(m) = 2(1-\varepsilon)^{(m-1)} \, S_0^2(m) \left[1-S_0^2(m)\right] \qquad (2.69)$$

$$g_{5i}(m) = 0 \; ; \; i=1,\ldots,4 \qquad (2.70)$$

$$g_{64}(m) = (1-\varepsilon)^m S_0^1(m)\ f_D^1(m) \tag{2.71}$$

$$g_{65}(m) = (1-\varepsilon)^m S_1^1(m)\ f_D^0(m) \tag{2.72}$$

$$g_{66}(m) = (1-\varepsilon)^m \left[S_0^1(m)\ S_1^1(m)\ \delta_{m5} + f_D^0(m)\ f_D^1(m) \right] \tag{2.73}$$

$$g_{69}(m) = \varepsilon(1-\varepsilon)^{(m-1)} S_0^1(m)\ S_1^1(m) \tag{2.74}$$

$$g_{6i}(m) = 0\ ;\ i=1,2,3,7,8 \tag{2.75}$$

$$g_{77}(m) = (1-\varepsilon)\delta_{m1} \tag{2.76}$$

$$g_{79}(m) = \varepsilon\ \delta_{m1} \tag{2.77}$$

$$g_{7i}(m) = 0\ ;\ i\neq 7,9 \tag{2.78}$$

$$g_{88}(m) = \delta_{m1} \tag{2.79}$$

$$g_{8i}(m) = 0\ ;\ i\neq 8 \tag{2.80}$$

$$g_{99}(m) = \delta_{m1} \tag{2.81}$$

$$g_{9i}(m) = 0\ ;\ i\neq 9 \tag{2.82}$$

The numerical values used for $S_i^j(m)$ in computing the core matrix for the dual redundant system considered in the example are given below.

$S_0^1(1)=0.9895$; $S_0^1(2)=0.9792$; $S_0^1(3)=0.9696$; $S_0^1(4)=0.9608$; $S_0^1(5)=0.9531$

$S_1^1(1)=0.7862$; $S_1^1(2)=0.5845$; $S_1^1(3)=0.3861$; $S_1^1(4)=0.2012$; $S_1^1(5)=0.0199$

$S_0^2(1)=0.7900$; $S_0^2(2)=0.5845$; $S_0^2(3)=0.3819$; $S_0^2(4)=0.1867$; $S_0^2(5)=0.0008$

$S_1^2(1)=0.7812$; $S_1^2(2)=0.5745$; $S_1^2(3)=0.3721$; $S_1^2(4)=0.1842$; $S_1^2(5)=0.0009$

The decision time mass functions considered for the SRDT's give $P_{fa}=0.047$ and those of the SPRT's give $P_{fa}=0.047$ and $P_m=0.02$.

2.3.4 Decomposition of the model

The semi-Markov model for the dual redundant fault tolerant system exhibits fast and slow transitions between various states and can be decomposed into different classes such that the transitions within each class are all in the fast time scale. Class 1 comprises states 1,2, and 3 (1 and 2 transient), class 2 states 4,5,6,7 and 8 (4,5 and 6 transient), and class 3 state 9. Class 2 contains two trapping states for $\varepsilon=0$ and hence is a non-ergodic class.

2.4 Modified ALgorithm for Generating Numerical Results

Earlier results based on Korolyuk's limit theorem for semi-Markov chains approximate the aggregated state model after time scale decomposition by a homogeneous Markov chain. The inter-class transition rates are derived from (among other things) the invariant distribution in each class.

Many states in a FTCS model lacking on-line repair are transient. When these transient states have large holding times, considerable error in the asymptotic approximation for the original semi-Markov model results over the time scales of interest. Furthermore, in some cases, occupation of these states is the only means by which some of the interclass transitions can occur. In these cases, the class probability approcimations can also be in considerable error.

One of the key results of the research during the period of the grant is the development of a modified algorithm to account for inter-class transitions from transient states with long holding times. In this algorithm, a non-homogeneous aggregated Markov chain is used to evaluate the class probabilities after decomposition. The time varying transition rates for this Markov chain are derived taking into account contributions from the transient states. This leads to a more complex algorithm for approximating

the state probabilities, but considerable improvement in the accuracy of the approximations is the result. See Appendix A for details.

The modified algorithm was investigated by applying it to the dual-redundant FTCS model derived in the preceding section. Comparison of the numerical results from the two asymptotic approximations are discussed in appendix A. Since the nine state model considered in the example is representative of FTCS models, the results are quite encouraging.

It is interesting to note that when the earlier limit theorem is used, there is 100% error in the estimates of the occupancy probabilities for all the transient states and significant error in the class probabilities as well. However, using the modified algorithm mentioned above, the transient states are better approximated with class probability errors within 10%. This is a significant improvement over the earlier methods and makes this technique a valuable tool for evaluation of FTCS designs.

3. SUMMARY OF SIGNIFICANT FINDINGS AND FUTURE WORK

3.1 Significant Findings

The work during the grant period produced two key findings. They are:

1. The extension of the approoximate aggregation results to decomposed classes that contain multiple trapping states. (See section 2.2.)
2. The development of the modified algorithm that accounts for interclass transitions from transient states with long holding times. (See sections 2.3 and 2.4 and Appendix A.)

The first of these findings is significant because it implies that approximate aggregation can be applied to a broad range of semi_markov reliability models of fault tolerant systems. The second finding is significant because the modified algorithm, while introducing another source of error, leads to significantly more accurate numerical results for models that include transient states with very long holding times within the decomposed classes.

3.2 Future Work

A proposal has been submitted to AFOSR to continue this work. The focus of the continuation of the work is to extend the class of models to which the modified algorithm can be applied. Also, the proposed effort will begin to examine fault tolerant systems from a control system performance viewpoint, a viewpoint which has been secondary to our reliability evaluation work. Finally, we will attempt to examine more complex models of fault tolerant systems.

4. PERSONNEL

This research was conducted in its entirety at the University of Cincinnati under the direction of Bruce K. Walker, Associate Professor of Aerospace Engineering and Engineering Mechanics. He was assisted by Ramaswamy Srichander, a doctoral candidate in Aerospace Engineering and Engineering Mechanics, whose support was provided primarily by the grant.

5. PAPERS AND PRESENTATIONS

The following papers and presentations resulted either partly or wholly due to the research reported in this report:

1. R. Srichander and B.K. Walker, "An Approximate Algorithm for Evaluation of Semi-Markov Reliability Models," 1989 American Control Conference, Pittsburgh, pp. 2653-2659, June 1989.

2. R. Srichander and B.K. Walker, "On the Operating Characteristics of a One-Sided Sequential Test," submitted to Technometrics, April 1989.

In addition, the following papers that were developed form the research reported in [7] were published:

N.M. Wereley and B.K. Walker, "Approximate Evaluation of Semi-Markov Chain Reliability Models," 27th IEEE Conference on Decision and Control, Austin, TX, pp. 2322-2329, December 1988.

B.K. Walker, N.M. Wereley, R.H. Luppold, & E. Gai, "Effects of Redundancy Management on Reliability Modeling," IEEE Trans. on Reliability, vol. 38, no. 4, pp. 475-482, October 1989.

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APPENDIX A

AN APPROXIMATE ALGORITHM FOR EVALUATION OF SEMI-MARKOV RELIABILITY MODELS

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ABSTRACT

Reliability models of fault tolerant control systems (FTCS) are described by semi-Markov models when sequential tests are used for failure detection and identification (FDI). The transient analysis of these semi-Markov chains are of interest because the steady state behaviour is trivial. The relatively rare occurence of the failure events compared to the fast decisions by the FDI tests allow time-scale decomposition of these models. This leads to an aggregated state model which is approximately Markovian in character in the limit when the failure rates approach zero. An aggregate non-homogeneous Markov model is considered over the transient period of interest and an algorithm to compute the transition rates from the aggregated states is described. From the aggregated model, the state probability distribution of the original semi-Markov model can be derived by a disaggregation step.

1. INRODUCTION

A fault-tolerant system architecture is described by a redundant set of components and a redundancy management (RM) algorithm to reconfigure the system when failures occur. These failures typically occur at a relatively slow rate compared to the RM decision events. The RM decisions are made based on FDI tests on output signals of interest. Since the FDI tests operate in a noisy environment, to keep the decision errors low (typically of order 10^{-2} to 10^{-3} per test) these tests are often sequential in nature. Such sequential tests have non-exponential holding times before a decision is made. This gives rise to semi-Markov models for the FTCS reliability models [1].

A semi-Markov chain is characterised by a discrete set of states and an arbitrary distribution of the holding or sojourn time for each transition. The semi-Markov chain specialises to a Markov chain when the holding times are geometrically and identically distributed for all transitions exiting a particular state.

In any FTCS design the designer must choose the thresholds for the FDI tests. These thresholds govern the probability of false alarms (P_{fa}) and the probability of missed detection (P_m) for each execution of the test. To investigate whether the thresholds are acceptable, one needs efficient but computationally simple techniques to evaluate the reliability of the system over a desired mission time. Since such designs are iterative in nature, it is imperative that the computational scheme be as simple as possible. The approximate reliability evaluation of semi-Markov models proposed in this paper is designed specifically to address the above issue. The results derived in the sequel are applied to a representative FTCS architecture and the approximate results are compared with the exact results obtained by solving numerically for the interval transition probability matrix $\Phi(n)$.

Many methods exist for evaluating the steady state behaviour of semi-Markov chains [2]. However, most FTCS models contain one or more trapping states (such as the system loss state) and hence the steady state behaviour is trivial and not of interest. In order to evaluate the reliability of the FTCS over the time period of interest, the interval transition probability matrix $\Phi(n)$ of the transient semi-Markov model must be computed [2]. The computation of $\Phi(n)$ involves convolution sums and hence is memory and computation intensive [3,4]. The time step 'n' over which $\Phi(n)$ must be evaluated is often very large in an absolute sense but short in comparison with the mean time between failures (MTBF) of the components.

Other schemes for transient analysis based on time-scale decomposition of original model into fault-handling and fault-occurence sub-models have been proposed [5,6]. In [5] techniques are described for multi-processor systems where FDI tests are single sample tests that give rise to Markov models. In [6] the presence of non-exponential sojourn times in the states are considered and an extended stochastic Petri Net model is used for the fault-handling behaviour while a non-homogeneous Markov chain is used for possibly non-Poisson fault-occurence behaviour. However, the presence of false alarms in developing the fault handling sub-models is not taken into consideration.

In the technique proposed here we consider only constant failure rates but arbitrary sojourn times in the various states and also take into consideration the probability of false alarms.

The paper is organised as follows. In section 2, the previous work in this area and existing limit theorems for semi-Markov chains are discussed. In section 3, an algorithm for approximat-

ing the transient behaviour of the original model is developed. Section 4 describes the application of the proposed technique to a two component redundant system architecture that uses sequential tests for failure detection.

2. BACKGROUND

Earlier results have been based on Korolyuk's limit theorem for semi-Markov chains [7], which approximates an aggregated state model by a homogeneous Markov chain. The original semi-Markov model is decomposed into various classes where each class contains a group of states characterised by an identical *number of* failures. For instance, class 1 contains all states with no failures, class 2 contains all states with one failure, etc. The inter-class transitions take place in a slow time-scale at a rate of order ε, the failure rate of the components. It has been shown that the aggregate class-to-class transitions have exponential sojourn times when the embedded Markov chain in each class for $\varepsilon=0$ is ergodic [7]. In [8], the invariant distribution in each class of the aggregate model is used to derive class-to-class transition rates. One difficulty with such a technique for FTCS is that, for systems that lack on-line repair capability, many of the states in each class are transient. When the original semi-Markov model probability distribution vector is recovered, the estimates for the transient states are zero [4]. These states may have large holding times, which in the original model may be of the order of the mission time, and hence approximating their probabilities by zero may not be valid. Futhermore, in some cases, occupation of these states is the only means by which some of the interclass transitions can occur. In these cases, class probabilities can also be considerably in error.

Another drawback of the technique in [8] is that the approximation is valid only if the embedded Markov chain in each class is ergodic. This means that each class must contain exactly one irreducible closed subset of positive persistent aperiodic states [9]. Many FTCS architectures give rise to non-ergodic classes when the time-scale decomposition is used. A modified algorithm is presented in [4] that relaxes this condition for some of the classes.

In section 3, we present an algorithm that gives good approximations for state probability distributions of semi-Markov models and also relaxes ergodicity condition for all classes.

First, we summarize the results in [8] and introduce notation used in the later sections.

2.1 Limit Theorem For Semi-Markov Chains

Let the set E of states of the semi-Markov chain be expressible as a union of disjoint classes:

$$E=\sum_{k=1}^{R} E_k \qquad k\in M=\left\{1,2,..,R\right\} \tag{1}$$

Let $\tau_{kr}^{(i)}$ be the sojourn time of the semi-Markov chain in class E_k when it starts from state $i\in E_k$ and moves to class E_r where $r\neq k$. Suppose the following two conditions hold for the semi-Markov chain E:

1. The elements of the core matrix sequence $\left\{g_{ij}^{\varepsilon}(n) \mid i,j\in E\right\}$ specifying the semi-Markov chain depend as follows on the small parameter ε:

$$g_{ij}^{\varepsilon}(n) = p_{ij}^{\varepsilon}\, h_{ij}\left(\frac{n}{\varepsilon}\right) \tag{2}$$

where $h_{ij}(n)$ is the holding time mass function for a transition from state i to state j and $h_{ij}(0)=0$. The p_{ij}^{ε} can be expanded in a Taylor series about $\varepsilon=0$. Retaining terms that are linear in ε:

$$p_{ij}^{\varepsilon}=\begin{cases} p_{ij}^{(k)}-\varepsilon q_{ij}^{(k)}+O(\varepsilon) \text{ if } i,j\in E_k \\ \varepsilon q_{ij}^{(k)}+O(\varepsilon) \text{ if } i\in E_k \text{and } j\notin E_k \end{cases} \tag{3}$$

The embedded Markov chain for $\varepsilon=0$ obeys the usual Markov chain properties:

$$\sum_{j\in E_k} p_{ij}^{(k)}=1; \text{ and } p_{ij}^{(k)}\in[0,1]; \ \forall\ k\in M \tag{4}$$

Here $\varepsilon q_{ij}^{(k)}$, $i,j\in E_k$ are probabilities by which the Markov chain defined by $\left[p_{ij}^{(k)}\right]$ is defective if the ε-dependent transitions are taken into account, and $\varepsilon q_{ij}^{(k)}$, $i\in E_k$, $j\notin E_k$ are ε-dependent out of class transition probabilities.

2. The embedded Markov chains defined by the matrices $\left\{p_{ij}^{(k)} \mid i,j\in E_k\ \forall\ k\in M\right\}$ are ergodic with stationary distribution $\left\{\pi_{iS}^{(k)} \mid i\in E_k\ \forall\ k\in M\right\}$.

Then:

$$\lim_{\varepsilon\ 0} Pr\left\{\tau_{kr}\le t\right\}=\gamma_{kr}\left\{1-\exp(-\lambda_k t/T)\right\} \tag{5}$$

where:

$$\gamma_{kr}= \frac{\sum_{i\in E_k} \pi_{iS}^{(k)} q_i^{(kr)}}{\sum_{i\in E_k} \pi_{iS}^{(k)} q_i^{(k)}} \tag{6}$$

$$\lambda_k= \frac{\sum_{i\in E_k} \pi_{iS}^{(k)} q_i^{(k)}}{\sum_{i\in E_k} \pi_{iS}^{(k)} \tau_i^{(k)}} \tag{7}$$

Here:

$$q_i^{(kr)} \equiv \sum_{j\in E_r} q_{ij}^{(k)} \tag{8}$$

$$q_i^{(k)} \equiv \sum_{j\in E_k} q_{ij}^{(k)} \tag{9}$$

$$\tau_i^{(k)} \equiv \sum_{j\in E_k} p_{ij}^{(k)} \bar{\tau}_{ij} \tag{10}$$

$$\bar{\tau}_{ij} \equiv \sum_{n=0}^{\infty} n\, h_{ij}(n) \tag{11}$$

For proof, we refer the reader to [4].

If we consider a discrete state continuous time semi-Markov chain with probability transition matrix $[p_{ij}^{\varepsilon}(t)]$, $i,j\in E$, and holding time cumulative distribution functions $F_{ij}(t/\varepsilon)$ depending on the small parameter ε, then identical results can be derived [4]. We now proceed to derive our algorithm for approximating the state probability vector of the original semi-Markov model.

3. MODIFIED ALGORITHM

An interesting aspect of many FTCS without on-line repair capability is that the inter-class transitions take place in an hierarchical manner with transitions from each class leading to a more degraded class, with the last class a trapping class. This implies that:

$$q_i^{(kr)} = q_i^{(k)} \quad \forall\ i\in E_k \quad \text{if} \quad k \longrightarrow r \tag{12}$$

and,

$$\gamma_{kr} = \begin{cases} 1 & \text{if } k \longrightarrow r \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $k \longrightarrow r$ implies that there exist transitions from class k to class r.

In [4] the invariant distribution in each class is used to compute the inter-class transition rates λ_k. It is clear from the results in [4] that the transition rates from the classes ultimately converge to this after the invariant distribution in each class is established, but may be different over a transient period that can be quite large. Hence, it is intuitively clear that if we are in a position to characterise a time varying λ_k for the transitions, we can get better approximations for the results. One way to achieve this is to define a time-varying $\lambda_k(k)$ that depends on the probability distribution over the transient period in each class so that out of class transitions from the transient states are given due importance. This will give rise to a non-homogeneous Markov chain for the aggregate model. To reduce the complexity of solving a non-homogeneous Markov chain, we assume transition rates are stepwise constant over the desired time interval.

We first define the notion of step-size for the algorithm that we will discuss later. Let 'm' be the desired time step at which the reliability of the system is to be evaluated. We sub-divide m into N intervals each of size m/N. We refer to the time interval m/N as the step-size of the algorithm. Notice that if the holding time in transient states in each class are very small compared to the time interval m, then the approximation in [4] of using constant λ_k over the entire interval is valid. The violation of this assumption in most FTCS models is the motivation behind our modified algorithm.

As discussed above, to incorporate a time-varying transition rate we need to evaluate the probability distribution in each class at various time steps for the case $\varepsilon=0$. In trying to do this, we are faced with the difficulty that the various classes can be semi-Markov and hence this probability distribution can depend upon the time at which the class-to-class transitions occur, which is not known. Apart from this, solution of semi-Markov models in each class is not attractive even though solution of such reduced order models is computationally feasible. An engineering approximation to overcome this drawback is to solve the reduced order semi-Markov model for class 1 and then compute the ε-dependant transition probabilities to class 2 to determine the probability distribution in this class. Making use of the assumed hierarchical nature of the FTCS model, we then proceed to compute the distributions in other classes in the same manner once the probability distribution in the previous classes is known.

The algorithm to compute the approximate probability distribution vector at the desired time step m is given below.

3.1 Aggregation step

Let $n \equiv m/N$ be the step-size of the algorithm and let $k/n=1,2,\ldots,N$. Let $\pi_i^{(1)}(k)$ be the probability of occupying the i^{th} state within class 1 when $\varepsilon=0$. As pointed out earlier, this can be obtained by solving the transient semi-Markov model of class 1. The approximate probability distribution in other classes are computed as follows.

For $k\in S\equiv\{n,2n,\ldots,Nn\}$ and $\nu=1,2,\ldots,R-1$

$$\beta_i^{(\nu+1)}(k) = \sum_{j\in E_\nu} \pi_j^{(\nu)}(k)\, q_{ji}/\bar{\tau}_{ji}\,;\ i\in E_{\nu+1} \tag{14}$$

$$\zeta_i^{(\nu+1)}(k) = \beta_i^{(\nu+1)}(k) \Big/ \sum_{i\in E_\nu} \beta_i^{(\nu+1)}(k) \tag{15}$$

$$\xi_i^{(\nu+1)}(k) = \xi_i^{(\nu+1)}(k-n) + \sum_{j\in E_{\nu+1}} \zeta_j^{(\nu+1)}(k-n) * (p\nu)_{ji} + \zeta_i^{(\nu+1)}(k) - \zeta_i^{(\nu+1)}(k-n) \tag{16}$$

$$\pi_i^{(\nu+1)}(k)=\xi_i^{(\nu+1)}(k)/\sum_{\iota\in E_{\nu+1}} \xi_i^{(\nu+1)}(k) \quad (17)$$

where $(p\nu)_{ji}$ is $(j,i)^{th}$ element of the matrix $[p_{ji}^{(\nu+1)}]^{m/N}$ and the initial conditions are:

$$\xi_i^{(\nu+1)}(0)=0 \quad (18)$$

$$\zeta_i^{(\nu+1)}(0)=0 \quad (19)$$

The new transition rates of the non-homogeneous Markov chain are determined as follows:

$$\lambda_\nu^n(k)=.5*\left[\lambda_\nu(k)+\lambda_\nu(m)\right]; \ \forall k\epsilon S \ \& \ \nu\epsilon M \quad (20)$$

where $\lambda_\nu(k)$ is evaluated as in (7) with $\pi_{iS}^{(k)}$ replaced by $\pi_i^{(\nu)}(k)$. Notice that for $\nu\neq 1$, we need not store $\pi_i^{(\nu)}(k)$ $\forall k$ if we compute at each step, $\lambda_\nu(k)$ $\forall\nu\epsilon M$. The averaging done in equation (20) gives a simple heuristic algorithm to account for transitions from transient states during the mission time of interest. The probability transition matrix at the desired time step 'm' is then determined from,

$$P(m)=\exp(A_k^{*}*\epsilon*n) \quad (21)$$

where,

$$A_k^{*}=\sum_{k=1}^{N} A_k \quad (22)$$

Here, A_k is the transition rate matrix of the Markov chain [2], given by,

$$A_k=\Lambda(k)*(\Gamma-I) \quad (23)$$

where $\Lambda(k)$ is a diagonal matrix with elements $\lambda_\nu^n(k)$, $\Gamma=[\gamma_{\nu r}]$ and I is the identity matrix.

3.2 Disaggregation step

From the approximate probability transition matrix P(m) the probability distribution vector of the aggregate model at the desired time step can be determined from the known initial condition. We denote its elements by $\rho_\iota(m)$, $\iota\epsilon M$. The state probability distribution of the original model are approximated at the disaggregation step as follows:

$$\eta_j^{(\iota)}(m)=\rho_\iota(m)*\pi_j^{(\iota)}(m); \ \forall\iota\epsilon M \ \& \ \forall j\epsilon E_\iota \quad (24)$$

$$\Pi_j^{(\iota)}(m)=\eta_j^{(\iota)}(m)/\sum_{\iota,j}\eta_j^{(\iota)}(m) \quad (25)$$

where $\Pi_j^{(\iota)}(m)$ are the elements of the probability distribution vector of the original semi-Markov chain.

4. NUMERICAL EXAMPLE

The proposed technique is applied to a two component redundant system that uses sequential tests for FDI. At the two level stage where both instruments are operational, two one-sided sequential ratio detection tests (SRDT) are used to detect a failure [1]. If the SRDTs simultaneously arrive at a decision, the test is reset. Depending on whether SRDT 1 or SRDT 2 arrives at a decision, an isolation option of either 1 or 2 is triggered. In either option, two sequential probability ratio tests (SPRT) are used to arrive at an isolation decision or an alarm rejection. Once an instrument is declared faulty, the FDI tests are discontinued. To simplify the calculation of the exact results required for comparison, it is assumed that tests are reset after five time samples, which reduces the number of terms in the numerical convolution sums to be evaluated to a maximum of five.

This strategy gives rise to a model with nine states:

1. *Two instruments available, none failed, no detection alarm present, SRDTs operating. (2/O/O)*
2. *Two instruments available, none failed, one SRDT detection alarm present, SPRTs operating. (2/O/D)*
3. *One instrument available, one eliminated due to false isolation [FDI discontinued]. (1G/FI)*
4. *Two instruments available, one failed, correct detection alarm triggered, SPRTs operating. (2/F/C)*
5. *Two instruments available, one failed, detection alarm present for wrong pair, SPRTs operating. (2/F/W)*
6. *Two instruments available, one failed, no detection alarms present, SRDTs operating. (2/F/O)*
7. *One good instrument available, one faulty instrument isolated [FDI discontinued]. (1G/F)*
8. *System loss due to one failure and one false isolation. (SL/F/FI)*
9. *System loss due to two failures. (SL/2F)*

The state transition diagram for this model is shown in Fig. 1. The thick lines indicate fast transitions while the dashed lines indicate slow transitions. When behavioural decomposition of the nine state model is done, it breaks up into three classes with the inter-class transitions taking place in a slow time-scale. Class 1 comprises states 1, 2, and 3 (1 & 2 transient), class 2 states 4,5,6,7 & 8 (4,5 & 6 transient), and class 3 state 9. Class 2 contains two trapping states for ϵ=0 and hence is a non-ergodic class. The decision time mass functions for the SRDTs were assumed to give P_{fa}=.047 and those of SPRTs to give P_{fa}=.047 and P_m=.02. The holding time mass functions $h_{ij}(n)$ are computed from the core matrix elements $g_{ij}^{\epsilon}(n)$ which are derived as indicated in [3]. The matrices $[p_{ij}]$, $[q_{ij}]$, and

$[\bar{\tau}_{ij}]$ were derived for two cases:

case(1): $\varepsilon = 5\times10^{-5}$

case(2): $\varepsilon = 5\times10^{-8}$

The matrices for *case(2)* are given in table 1 truncated to 4 decimal places.

We define normalized error between the exact and approximate class probabilities as follows:

$$\text{Normalized error}=\frac{\pi^{(\nu)}_{\text{exact}} - \pi^{(\nu)}_{\text{approx}}}{\pi^{(\nu)}_{\text{exact}}} \quad (26)$$

where $\pi^{(\nu)}_{\text{exact}}$ is the exact probability distribution in class $\nu \in M$ calculated numerically. The normalized error for different class probabilities using the above algorithm for both cases considered is shown in Fig. 2-7. We notice that the normalized error in all the classes is less than 0.1 (except in one case) at the desired time step of interest and the error decreases as the step-size decreases.

A comparison of the exact state probabilities at the desired time step with the approximate probabilities derived from (i) the algorithm presented here and (ii) the algorithm given in [8] is shown in table 2. A modified version of the algorithm presented in [4] to treat non-ergodic classes is used to derive the approximate probabilities in case (ii).

4.1 Discussion of results

From table 2, we infer that there is always 100% error between the exact and approximate probabilities for transient states using the algorithm given in [4]. Also, the class 3 probability is always approximated as zero implying a 100% error also exists in one of the class probabilities. The reason for this is that the transient states in class 1 have large holding times that are of the order of the mission time of interest, whereas in [4] it is assumed implicitly that holding times are negligible relative to the mission time. Such holding time mass functions are not unusual in FTCS design when we try to keep the probability of a false alarm and a false isolation very small, as is desirable.

The numerical example presented here is typical of many FTCS architectures. The good approximation of the results indicates the usefulness of the algorithm for approximate reliability evaluation. Comparing the exact state probabilities with those computed using the approximate technique presented here, we infer that the results agree well for small ε over large mission times which are still considerably smaller than the MTBF of the components.

However, from Fig. 2-7 it is noticed that, although the normalized error becomes smaller as the step-size decreases, the convergence of the algorithm is not asymptotic. Hence, an optimum choice for the step-size is not clear. A good choice depends on the mission time of interest, and the step-size can be large for large mission times. One rule of thumb is to choose n=50 to 100. This gave good approximations in most cases investigated in this work.

5. CONCLUSION

An algorithm for approximate evaluation of the state probability vector of a semi-Markov process governed by widely different transition rates has been presented. The method was used to evaluate the reliability of a representative FTCS architecture and the superiority of the scheme over that in [4] has been demonstrated. Also, the scheme is applicable to systems that contain classes with more than one recurrent chain when the behavioural decomposition is used. It is noticed that the approximation is better when the time scales are distinct and the step-size of the algorithm is small. The sensitivity of the scheme to the time scale separation and order of the failure rate ε remains to be investigated.

The primary contribution of the paper is the extension to more general classes of systems that cannot be handled by the scheme in [4,8] and better approximation of the transient state probabilities of the original model that might have large holding times.

ACKNOWLEDGEMENT

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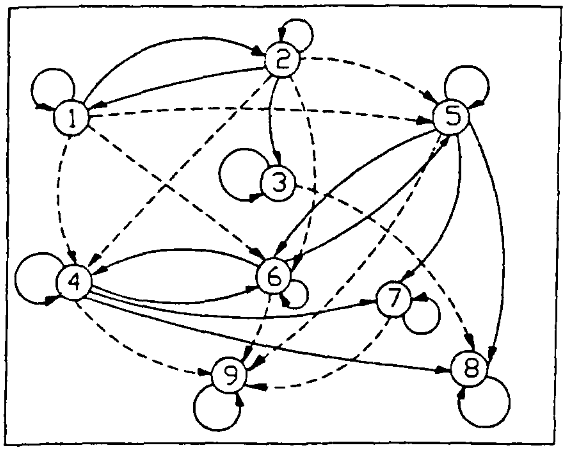

Fig. 1. State transition diagram for nine state model

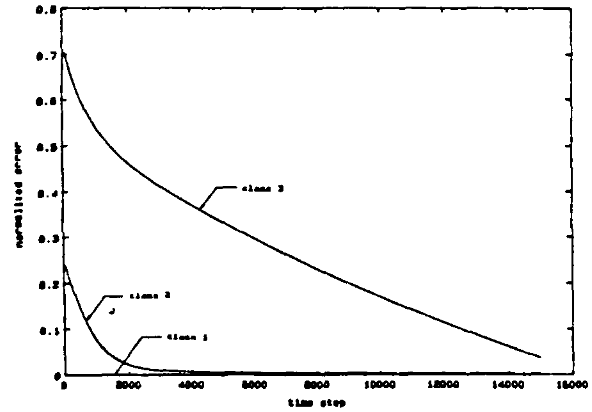
Fig. 2. Normalized error for $\varepsilon=5\times10^{-8}$, n=50 & m=15,000

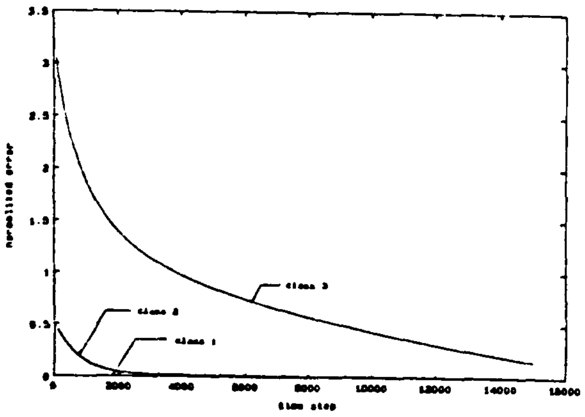
Fig. 3. Normalized error for $\varepsilon=5\times10^{-8}$, n=100 & m=15,000

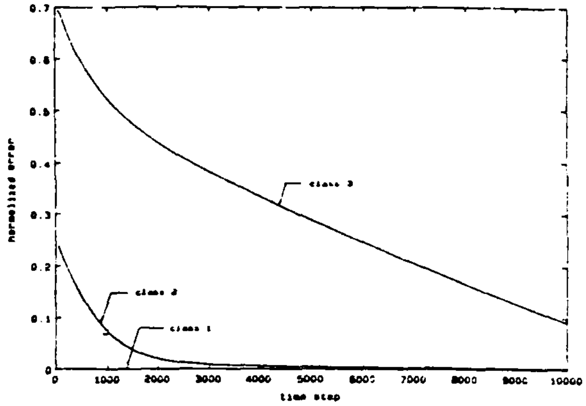
Fig. 4. Normalized error for $\varepsilon=5\times10^{-8}$, n=50 & m=10,000

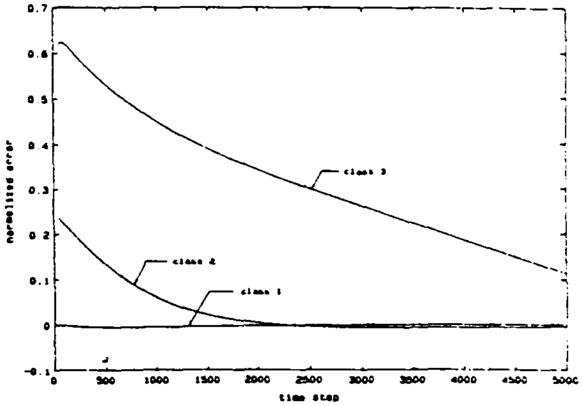
Fig. 5. Normalized error for $\varepsilon=5\times10^{-5}$, n=50 & m=5,000

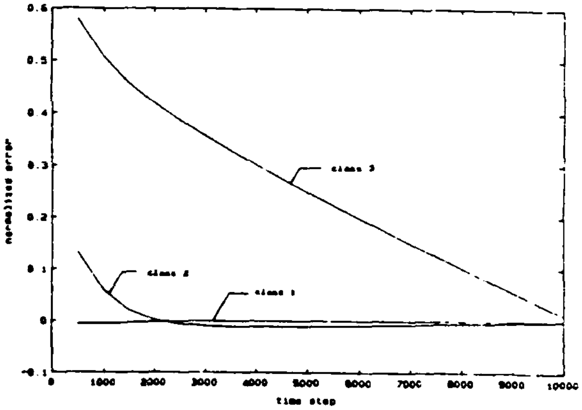
Fig. 6. Normalized error for $\varepsilon=5\times10^{-5}$, n=50 & m=10,000

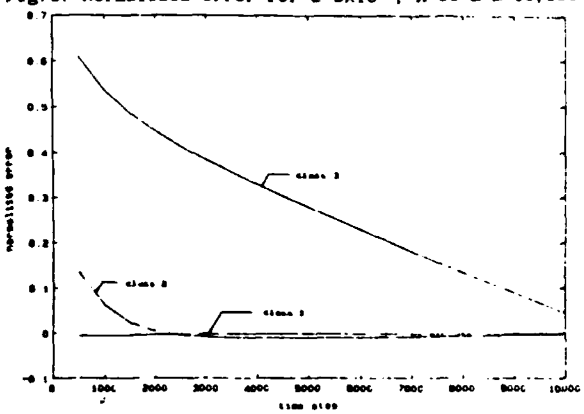
Fig. 7. Normalized error for $\varepsilon=5\times10^{-5}$, n=100 & m=10,000

Table 1

$$[p_{ij}] = \begin{bmatrix} 0.9087 & 0.0913 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.9067 & 0.0039 & 0.0894 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0478 & 0.0 & 0.0181 & 0.9332 & 0.0009 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0039 & 0.9067 & 0.0447 & 0.0447 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.9519 & 0.0199 & 0.0282 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$[q_{ij}] = \begin{bmatrix} 9.0018 & 0.5400 & 0.0 & 0.0913 & 0.0913 & 9.4192 & 0.0 & 0.0 & 0.0 \\ 5.4293 & 0.0271 & 0.5449 & 2.5000 & 2.5000 & 0.9067 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0689 & 0.0 & 0.0232 & 1.1869 & 0.0013 & 1.5980 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0082 & 1.6453 & 0.0436 & 0.0436 & 1.5951 \\ 0.0 & 0.0 & 0.0 & 1.6481 & 0.0189 & 0.0918 & 0.0 & 0.0 & 1.9370 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$[\bar{\tau}_{ij}] = \begin{bmatrix} 4.9989 & 2.8341 & 0.0 & 2.8341 & 2.8341 & 2.9623 & 0.0 & 0.0 & 0.0 \\ 3.7424 & 4.2362 & 3.6921 & 3.0 & 3.0 & 3.7424 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.7270 & 0.0 & 3.5445 & 3.7294 & 3.4850 & 2.4646 \\ 0.0 & 0.0 & 0.0 & 0.0 & 4.2362 & 3.7424 & 3.6921 & 3.6921 & 2.4721 \\ 0.0 & 0.0 & 0.0 & 2.8977 & 1.9723 & 4.2688 & 0.0 & 0.0 & 2.0206 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.\text{[illegible]} & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

Table 2
Comparison of exact and approximate probabilities

Time step and ε	Exact results (state 1 to 9)	Approx. results using step-size=50	Approx. results using scheme [4]
10,000	0.000027684938	0.000028037752	0
	0.000009194938	0.000008243412	0
$\varepsilon=5*10^{-8}$	0.999432770000	0.999434580306	0.999500124979
	0.000000000047	0.000000000021	0
	0.000000000003	0.000000000021	0
	0.000000000156	0.000000000241	0
	0.000061669460	0.000054278419	0
	0.000468653400	0.000474833582	0.000499875021
	0.000000028905	0.000000026245	0
15,000	0.000027582208	0.000027927991	0
	0.000009187481	0.000008233990	0
$\varepsilon=5*10^{-8}$	0.999183050000	0.999184825377	0.999250281179
	0.000000000057	0.000000000020	0
	0.000000000003	0.000000000020	0
	0.000000000156	0.000000000236	0
	0.000061654051	0.000053300697	0
	0.000718480390	0.000725668978	0.000749718820
	0.000000044321	0.000000042689	0
10,000	0.000033414812	0.000016518806	0
	0.000835702550	0.000016518806	0
$\varepsilon=5*10^{-5}$	0.588035090000	0.588829863265	0.606530659712
	0.000006652044	0.000000015147	0
	0.000000329155	0.000000015147	0
	0.000003095837	0.000000177613	0
	0.036292407000	0.039965861579	0
	0.353017460000	0.349653693105	0.393436934028
	0.021775854000	0.021528998622	0